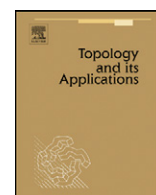




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On productively Lindelöf spaces

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ABSTRACT

The class of spaces such that their product with every Lindelöf space is Lindelöf is not well-understood. We prove a number of new results concerning such productively Lindelöf spaces with some extra property, mainly assuming the Continuum Hypothesis.

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1. Applications of elementary submodels

A quick introduction to the method of elementary submodels in our context is given in Appendix A.

Definition 1.1. A topological space is *productively Lindelöf* if its product with every Lindelöf space is Lindelöf. A space is *powerfully Lindelöf* if its ω -th power is Lindelöf.

Problem 1.2. (Michael [27,28]) Are productively Lindelöf spaces powerfully Lindelöf?

Lemma 1.3. (Alster [2]) The Continuum Hypothesis implies productively Lindelöf T_3 spaces of weight $\leq \aleph_1$ are powerfully Lindelöf.

Since Lindelöf first countable T_2 spaces have cardinality (and hence weight) at most continuum, we see that, assuming the Continuum Hypothesis, productively Lindelöf first countable T_3 spaces are powerfully Lindelöf [9]. This can be extended, as follows.

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Theorem 1.4. *The Continuum Hypothesis implies that productively Lindelöf sequential T_3 spaces are powerfully Lindelöf.*

Proof. Let X be productively Lindelöf sequential T_3 , and \mathcal{U} be an open cover of X^ω . Without loss of generality, \mathcal{U} is composed of basic open subsets of X^ω . Let M be a countably closed elementary submodel of H_θ of size $2^{\aleph_0} = \aleph_1$, for θ regular and sufficiently large, such that M contains X, \mathcal{U} , and anything else needed. For any space $Y \in M$, let Y_M be the topology on $Y \cap M$ generated by the sets $U \cap M$ where $U \in M$ is open in Y .

Since M is countably closed, $X \cap M$ is a closed subset of X and thus also productively Lindelöf. Since every open set in X_M is open in $X \cap M$ with the relative topology, X_M is a continuous image of $X \cap M$, and therefore X_M is productively Lindelöf.² The weight of X_M is $\leq |M| \leq \aleph_1$, so by Alster's Lemma 1.3, $(X_M)^\omega$ is Lindelöf.

Since M is countably closed, $X^\omega \cap M = (X \cap M)^\omega$, that is, $(X^\omega)_M = (X_M)^\omega$ as sets. Also, for $B_i \in M$ open in X , the set $\prod_{i < \omega} B_i \subseteq X^\omega$, when intersected with M , is just $\prod_{i < \omega} (B_i \cap M)$, so we see that as spaces, $(X^\omega)_M = (X_M)^\omega$.

Thus, $(X^\omega)_M$ is Lindelöf. As $\mathcal{U} \in M$, we have by elementarity that $\{U \cap M : U \in \mathcal{U} \cap M\}$ is an open cover of $(X^\omega)_M$. Thus, there are $\{U_n : n < \omega\} \subseteq \mathcal{U} \cap M$ such that $\{U_n \cap M : n < \omega\}$ covers $(X^\omega)_M$. Since M is countably closed, $\{U_n : n < \omega\} \in M$. $M \models \{U_n : n < \omega\}$ covers X^ω , so indeed $\{U_n : n < \omega\}$ covers X^ω . \square

It would be nice to eliminate the hypothesis that X be sequential. This was only used to get that the sequentially closed set $X \cap M$ is indeed closed. There are large compact (hence productively Lindelöf) sequential T_3 spaces, so “sequential” is indeed an improvement over “first countable”. We do not know whether the Continuum Hypothesis is necessary for these results.

It is not known whether the weight restriction in Alster's Lemma 1.3 can be removed, nor whether the Continuum Hypothesis is necessary. There is no reason to believe the weight of a productively Lindelöf space cannot exceed its cardinality, so the following result is not obvious.

Corollary 1.5. *The Continuum Hypothesis implies productively Lindelöf T_3 spaces of cardinality \aleph_1 are powerfully Lindelöf.*

Proof. As in the proof of Theorem 1.4, we take a countably closed elementary submodel M of H_θ with $|M| = \aleph_1$, such that M contains everything needed, and, in addition, $X \subseteq M$. It follows that the weight of X_M is $\leq \aleph_1$. Since $X \cap M = X$, X_M is a continuous image of X and hence is productively Lindelöf and so powerfully Lindelöf, and hence, as before, X is powerfully Lindelöf. \square

2. Selective covering properties

Definition 2.1. A *point-cofinite cover* of a space is an infinite open cover such that each point is in all but finitely many members of the cover.

Definition 2.2. A topological space X is:

- (1) *Alster* if each cover of X by G_δ sets such that each compact set is included in one of them has a countable subcover.³
- (2) *Hurewicz* if for each sequence $\{\mathcal{U}_n\}_{n < \omega}$ of open covers without finite subcovers, there are finite $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\{\bigcup \mathcal{F}_n : n < \omega\}$ is a point-cofinite cover.
- (3) *Menger* if for each sequence $\{\mathcal{U}_n\}_{n < \omega}$ of open covers without finite subcovers, there are finite $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\{\bigcup \mathcal{F}_n : n < \omega\}$ is a cover.

Let P be a property of topological spaces. A space X is *powerfully P* if X^ω has the property P . X is *finitely powerfully P* if all its finite powers of X have the property P .

By the definition, Hurewicz spaces are Menger. Alster spaces are productively and powerfully Lindelöf [2]. A slightly extended version of an argument from [9] yields the following generalization of results from [9,31].

Theorem 2.3. *Alster spaces are Hurewicz.*

Proof. Let $\{\mathcal{U}_n\}_{n < \omega}$ be a sequence of open covers of X without finite subcovers. We may assume that each \mathcal{U}_n is closed under finite unions. Let

$$\mathcal{U} = \left\{ \bigcap_{n < \omega} U_n : \forall n, U_n \in \mathcal{U}_n \right\}.$$

² A general form of this argument appears in Junqueira and Tall [24], see Proposition A.3 in Appendix A.

³ Alster's terminology [2] is slightly different, but equivalent.

Since X is Alster, there is a countable subcover $\{V_m: m < \omega\}$ of \mathcal{U} . For each m , write $V_m = \bigcap_{n < \omega} U_{mn}$, where $U_{mn} \in \mathcal{U}_n$ for all n . Then for each $x \in X$, $x \in \bigcup_{m \leq n} U_{mn}$ for all but finitely many n . \square

Thus, each property in Definition 2.2 implies the next one. These implications are strict: For sets of reals (indeed, for arbitrary spaces where every compact set is G_δ), Alster is clearly equivalent to σ -compact, and Hurewicz fits strictly between σ -compact and Menger.⁴

Corollary 2.4. *Alster spaces are finitely powerfully Hurewicz.*

Proof. Finite products of Alster spaces are Alster [2,10]. The proof in [10] does not use separation axioms. Apply Theorem 2.3. \square

A set of reals is *totally imperfect* if it includes no uncountable perfect (equivalently, compact) set.

Theorem 2.5. *There is a finitely powerfully Hurewicz set of reals which is not productively Lindelöf (and hence not Alster).*

Proof. Michael [26] proved that a totally imperfect set of reals is not productively Lindelöf. Bartoszyński and the second named author [11] proved that there is a totally imperfect, finitely powerfully Hurewicz set of reals. \square

Lemma 2.6. *(Alster [2]) The Continuum Hypothesis implies productively Lindelöf T_3 spaces of weight $\leq \aleph_1$ are Alster.*

Alster asked whether every productively Lindelöf space is Alster [2]. Alster's problem is still open. The following problem may be easier.

Problem 2.7. Does the Continuum Hypothesis imply productively Lindelöf sequential T_3 spaces are Alster?

Theorem 2.8. *The Continuum Hypothesis implies productively Lindelöf sequential T_3 spaces are finitely powerfully Hurewicz.*

Proof. It suffices to show that if $(X^k)_M$ is Hurewicz, then X^k is Hurewicz. We assume without loss of generality that the sequence $\{\mathcal{U}_n\}_{n < \omega}$ of open covers is in M . Then for each n , there is a finite $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup\{U \cap M: U \in \mathcal{F}_n\}$ is a point-cofinite cover of $(X^k)_M$. Note that each $\mathcal{U}_n \in M$. Since \mathcal{U}_n is countable, it is included in M , and hence each $\mathcal{F}_n \in M$. As M is countably closed, $\{\mathcal{F}_n\}_{n < \omega} \in M$. Thus,

$$M \models \left\{ \bigcup \mathcal{F}_n: n < \omega \right\} \text{ is a point-cofinite cover of } X^k,$$

and therefore the same holds in “the real world”, so indeed X^k is Hurewicz. \square

D-spaces were defined in [13]. See also [14,18].

Definition 2.9. A space X is *D* if for every *neighborhood assignment* $\{V_x\}_{x \in X}$ (i.e., each V_x is an open set containing x), there is a closed discrete $Y \subseteq X$ such that $\{V_x\}_{x \in Y}$ covers X .

Aurichi [8] proved that Menger spaces are D. Thus, assuming the Continuum Hypothesis, productively Lindelöf sequential T_3 spaces are finitely productively D. L. Zdomskyy pointed out to us that this last assertion can be generalized substantially. A *Michael space* is a Lindelöf space M such that $M \times \mathbb{P}$ (the space of irrationals) is not Lindelöf. Michael spaces can be constructed from a variety of axioms (in particular, from The Continuum Hypothesis), and it is a major open problem whether they can be constructed outright in ZFC. If there is a Michael space M , then productively Lindelöf spaces are Menger (and thus D) [30]. Indeed, Zdomskyy proves in [34] that if X is not Menger, then \mathbb{P} is a compact-valued upper-semicontinuous image of X . Thus, if X is not Menger, then the non-Lindelöf space $M \times \mathbb{P}$ is a compact-valued upper-semicontinuous image of $M \times X$. Consequently, $M \times X$ is not Lindelöf.

3. Indestructibly productively Lindelöf spaces

Definition 3.1. A space is *indestructibly productively Lindelöf* if it is productively Lindelöf in every countably closed forcing extension.

⁴ For an accessible exposition of this result, see [32].

Aurichi and the first named author proved that a metrizable space is indestructibly productively Lindelöf if and only if it is σ -compact [9]. It is easily seen that if a space Y is Hurewicz in a countably closed extension, then it is Hurewicz. The following theorem answers a question of Aurichi and the first named author [9].

Theorem 3.2. *Indestructibly productively Lindelöf T_3 spaces are powerfully Lindelöf and finitely powerfully Hurewicz (in particular, finitely powerfully D).*

Proof. Powerfully Lindelöf: Collapse $\max(2^{\aleph_0}, w(X))$ to \aleph_1 via countably closed forcing. In the extension, the indestructibly productively Lindelöf X remains productively Lindelöf and hence, by the Continuum Hypothesis, X becomes Alster. Then by Lemma 1.3, X becomes powerfully Lindelöf. But as a set, X^ω in the extension is the same as X^ω in the ground model. Since the space X^ω is Lindelöf in a countably closed extension, it is Lindelöf in the ground model, as claimed.

Finitely powerfully Hurewicz: In the extension obtained by collapsing as above, X is Alster, so every finite power of X is Hurewicz. But then every finite power of X is Hurewicz in the ground model. \square

Problem 3.3. Are indestructibly productively Lindelöf spaces Alster?

Corollary 3.4. *Indestructibly productively Lindelöf p -spaces are σ -compact.*

Proof. Let X be indestructibly productively Lindelöf and a p -space in the sense of Arhangel'skiĭ [4]. Then, as a paracompact p -space, X maps perfectly onto a metrizable Y . Let Z be Lindelöf in a countably closed extension. Then $X \times Z$ is Lindelöf. But then, by continuity, so is $Y \times Z$. So Y is indestructibly productively Lindelöf. But for metrizable spaces, indestructible productive Lindelöfness is equivalent to σ -compactness [9], which latter property is a perfect invariant. \square

Arhangel'skiĭ [6] proved that if X^ω is Lindelöf, then either X is compact or X^ω includes a closed copy of \mathbb{P} . Since the latter option is impossible for Menger spaces, he concluded that X is powerfully Menger if and only if X is compact.⁵

Corollary 3.5. *If there is a Michael space, then for every space X , X^ω is productively Lindelöf if and only if X is compact.* \square

Theorem 3.6. *If X^ω is indestructibly productively Lindelöf, then X is compact.*

Proof. Again, collapse $\max(2^{\aleph_0}, w(X))$ to \aleph_1 . In the extension, X^ω is productively Lindelöf and there is a Michael space, since the Continuum Hypothesis holds, which implies that there is a Michael space [26]. Therefore X is compact in the extension, and so is compact. \square

4. Mengerizing Michael's problems

As Menger implies Lindelöf, the classic problems about productively Lindelöf spaces make sense when *Lindelöf* is replaced by *Menger*.

Example 4.1. ω (the countable discrete space) is productively Menger, but not powerfully Menger.

The product of a Menger space with \mathbb{P} cannot be Menger since \mathbb{P} is not Menger, but the question whether the product of a Menger space with \mathbb{P} must be Lindelöf is less trivial. We will show that the answer is negative, in a very strong sense.

Definition 4.2. An open cover \mathcal{U} of a space X is an ω -cover if $X \notin \mathcal{U}$, but for each finite subset F of X there is $U \in \mathcal{U}$ containing F . X is a γ -space if each ω -cover of X includes a point-cofinite cover.

γ -spaces were introduced by Gerlits and Nagy [17], who proved that, for Tychonoff spaces, X is a γ -space if and only if the space $C_p(X)$ (the continuous the real-valued functions on X with the topology of pointwise convergence) is Fréchet–Urysohn. This is a very strong property. It is, for example, consistent that all metrizable γ -spaces are countable [17]. If X is a γ -space then X is Hurewicz. Being a γ -space is preserved by finite powers [17]. In particular, γ -spaces are finitely powerfully Hurewicz.

For $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n . A subset of ω^ω is *unbounded* if it is unbounded with respect to \leq^* . The minimal cardinality of an unbounded subset of ω^ω is denoted \mathfrak{b} . $\aleph_1 \leq \mathfrak{b} \leq 2^{\aleph_0}$. In particular, the Continuum Hypothesis implies $\mathfrak{b} = \aleph_1$. Additional information on \mathfrak{b} and similar combinatorial cardinal characteristics of the continuum can be found in [12].

⁵ Arhangel'skiĭ denotes by “Hurewicz” the property we call “Menger”. We use the currently accepted terminology.

We identify elements $x \in [\omega]^\omega$ with increasing elements of ω^ω by letting $x(n)$ be the n -th element of x . We will need the following well-known fact. For the reader's convenience, we reproduce here the proof given in [33].

Lemma 4.3 (Folklore). *If $B \subseteq [\omega]^\omega$ is unbounded, then for each increasing $f \in \omega^\omega$, there is $x \in B$ such that $x \cap (f(n), f(n+1)) = \emptyset$ for infinitely many n .*

Proof. Assume that f is a counterexample. Let g dominate all functions $f_m(n) = f(n+m)$, $m < \omega$. Then for each $x \in B$, $x \leq^* g$. Indeed, let m be such that for all $n \geq m$, $x \cap (f(n), f(n+1)) \neq \emptyset$. Then for each n , the n -th element of x is smaller than $f_{m+1}(n)$. \square

Orenshtein and the second named author [33] proved that an assumption weaker than $\mathfrak{b} = \aleph_1$ implies that there is an uncountable γ -space $X \subseteq \mathbb{R}$. The proof of the forthcoming Theorem 4.6 is a modification of their proof, slightly simplified in light of the stronger assumption.

Definition 4.4. Identify $P(\omega)$ with the Cantor space 2^ω , using characteristic functions. This defines the *Cantor topology* on $P(\omega)$. Consider the finer, *Michael topology* on $P(\omega)$ obtained by declaring all elements of $[\omega]^\omega$ isolated. Henceforth, unless otherwise indicated, $P(\omega)$ is always considered with the Michael topology.

The basic open sets in the Cantor topology of $P(\omega)$ are thus those of the form

$$[s, n] = \{x \in P(\omega) : x \cap \{0, \dots, n-1\} = s\},$$

where $n \in \omega$ and $s \subseteq \{0, \dots, n-1\}$. We will use the following modification of Lemma 1.2 of Galvin and Miller [16].

Lemma 4.5. *Consider $P(\omega)$ with the Michael topology. Assume that $[\omega]^{<\omega} \subseteq Y \subseteq P(\omega)$, Y is countable, and \mathcal{U} is a family of open subsets of $P(\omega)$ such that each finite subset of Y is included in some member of \mathcal{U} . There are $m_0 < m_1 < \dots$ and (not necessarily distinct) $U_0, U_1, \dots \in \mathcal{U}$ such that:*

- (1) *For each $y \in Y$, $y \in U_n$ for all but finitely many n .*
- (2) *For each $x \subseteq \omega$, $x \in U_n$ whenever $x \cap (m_n, m_{n+1}) = \emptyset$.*

Proof. Enumerate $Y = \{y_n : n < \omega\}$.

Let $m_0 = 0$. For each $n \geq 0$: Take $U_n \in \mathcal{U}$, such that $P(\{0, \dots, m_n\}) \cup \{y_0, \dots, y_n\} \subseteq U_n$. Let $s \subseteq \{0, \dots, m_n\}$. As $s \in [\omega]^{<\omega}$ and U_n is a neighborhood of s , U_n includes a neighborhood of s in the Cantor set topology, and thus there is k_s such that for each $x \in P(\omega)$ with $x \cap \{0, \dots, k_s - 1\} = s$, $x \in U_n$. Let $m_{n+1} = \max\{k_s : s \subseteq \{0, \dots, m_n\}\}$. \square

Since $P(\omega)$ is equipped with a topology finer than that of Cantor's space, which is metrizable, the following result cannot be proved outright in ZFC.

Theorem 4.6. *Consider $P(\omega)$ with the Michael topology. If $\mathfrak{b} = \aleph_1$, then there is a γ -space $X \subseteq P(\omega)$ which is also a Michael space (i.e., such that $X \times \mathbb{P}$ is not Lindelöf).*

Proof. For $x, y \in [\omega]^\omega$, $x \subseteq^* y$ means that $x \setminus y$ is finite. As $\mathfrak{b} = \aleph_1$, there is an unbounded (with respect to \subseteq^*) set $\{x_\alpha : \alpha < \aleph_1\} \subseteq [\omega]^\omega$ such that for all $\alpha < \beta < \aleph_1$, $x_\beta \subseteq^* x_\alpha$.

Let

$$X = \{x_\alpha : \alpha < \aleph_1\} \cup [\omega]^{<\omega} \subseteq P(\omega),$$

with the subspace topology (so that the elements x_α are isolated), and consider $X \times [\omega]^\omega$, where the space $[\omega]^\omega$ on the right is endowed with the ordinary Cantor space topology, so that it is homeomorphic to \mathbb{P} . The uncountable set $\{(x_\alpha, x_\alpha) : \alpha < \aleph_1\}$ is closed and discrete in $X \times [\omega]^\omega$. Thus, this space is not Lindelöf. Once we prove that X is a γ -space, we will have in particular that X is Lindelöf, so that X is a Michael space. That X is a Michael space is essentially proved in [12]; that X is a γ -space is new.

Let \mathcal{U} be an ω -cover of X . For each $\alpha < \aleph_1$, let $X_\alpha = \{x_\beta : \beta < \alpha\} \cup [\omega]^{<\omega}$.

Let $\alpha_0 = 0$. By Lemma 4.5, there are $m_0^0 < m_1^0 < \dots$ and elements $U_0^0, U_1^0, \dots \in \mathcal{U}$ such that each member of X_{α_0} is in U_n^0 for all but finitely many n , and for each $x \in P(\omega)$, $x \in U_n^0$ whenever $x \cap (m_n^0, m_{n+1}^0) = \emptyset$. Let $D_0 = \omega$.

As $\alpha_0 < \aleph_1$, $\{x_\alpha : \alpha_0 < \alpha < \aleph_1\}$ is unbounded. By Lemma 4.3, there is $\alpha_1 > \alpha_0$ such that $D_1 = \{n : x_{\alpha_1} \cap (m_n^0, m_{n+1}^0) = \emptyset\}$ is infinite. By Lemma 4.5, there are $m_0^1 < m_1^1 < \dots$ and members $U_0^1, U_1^1, \dots \in \mathcal{U}$ such that each member of X_{α_1} is in U_n^1 for all but finitely many n , and for each $x \in P(\omega)$, $x \in U_n^1$ whenever $x \cap (m_n^1, m_{n+1}^1) = \emptyset$.

Continue in the same manner to define, for each $k > 0$, elements with the following properties:

- (1) $\alpha_k > \alpha_{k-1}$;
- (2) $D_k = \{n: x_{\alpha_k} \cap (m_n^{k-1}, m_{n+1}^{k-1}) = \emptyset\}$ is infinite;
- (3) $m_0^k < m_1^k < \dots$;
- (4) $U_0^k, U_1^k, \dots \in \mathcal{U}$;
- (5) each member of X_{α_k} is in U_n^k for all but finitely many n ; and
- (6) for each $x \in P(\omega)$, $x \in U_n^k$ whenever $x \cap (m_n^k, m_{n+1}^k) = \emptyset$.

Let $\alpha = \sup_k \alpha_k$. Then $\alpha < \aleph_1$, X_α is countable, and $X_{\alpha_k} \subseteq X_{\alpha_{k+1}}$ for all k . Thus, there are for each k a finite $F_k \subseteq X_{\alpha_k}$ such that $F_k \subseteq F_{k+1}$ for all k , and $X_\alpha = \bigcup_k F_k$. For each k , let $I_k = \{n \in D_k: F_k \subseteq U_n^k\}$. I_k is an infinite (indeed, cofinite) subset of D_k , and for each $x \in X_\alpha$, if N is the first with $x \in F_N$, then x belongs to $\bigcap_{n \in I_k} U_n^k$ for all $k \geq N$.

Take $n_0 \in I_1$. For $k > 0$, take $n_k \in I_{k+1}$ such that $m_{n_k}^k > m_{n_{k-1}+1}^{k-1}$, $x_\alpha \cap (m_{n_k}^k, m_{n_k+1}^k) \subseteq x_{\alpha_{k+1}} \cap (m_{n_k}^k, m_{n_k+1}^k)$, and $U_{n_k}^k \notin \{U_{n_1}^1, \dots, U_{n_{k-1}}^{k-1}\}$. We claim that each member of X is in $U_{n_k}^k$ for all but finitely many k .⁶ By the last assertion in the previous paragraph, this is true for each member of X_α . As for each $\beta \geq \alpha$ we have that $x_\beta \subseteq^* x_\alpha$, it suffices to show that for each $x \subseteq^* x_\alpha$, $x \in U_{n_k}^k$ for all but finitely many k . For each large enough k , $m_{n_k}^k$ is large enough, so that

$$x \cap (m_{n_k}^k, m_{n_k+1}^k) \subseteq x_\alpha \cap (m_{n_k}^k, m_{n_k+1}^k) \subseteq x_{\alpha_{k+1}} \cap (m_{n_k}^k, m_{n_k+1}^k) = \emptyset,$$

since $n_k \in D_{k+1}$. Thus, $x \in U_{n_k}^k$. \square

5. Analytic spaces

The first named author proved in [31] that every analytic, metrizable, productively Lindelöf space is σ -compact if and only if there is a Michael space. The hypothesis of metrizability can be removed. According to Arhangel'skiĭ [6], a space is *analytic* if it is a continuous image of the space \mathbb{P} of irrationals.

Theorem 5.1. *Every analytic productively Lindelöf space is σ -compact if and only if there is a Michael space.*

Proof. By their definition, analytic spaces are Lindelöf. Relying on results of Jayne and Rogers [22], Arhangel'skiĭ [6] proved that analytic spaces are perfect pre-images of metrizable spaces. Since both productive Lindelöfness and σ -compactness are perfect invariants, we are done. \square

Perfect pre-images of analytic spaces are called *properly analytic* in [22]. It follows immediately that every properly analytic, productively Lindelöf space is σ -compact if and only if there is a Michael space.

According to Hansell [19], a space is *K-analytic* if it is the continuous image of a Lindelöf Čech-complete space.

Problem 5.2. Is it consistent that every productively Lindelöf *K*-analytic space is σ -compact?

The first named author also proved in [31] that the Axiom of Projective Determinacy implies that every projective, metrizable, productively Lindelöf space is σ -compact if and only if there is a Michael space.

We can certainly extend this to perfect pre-images of projective metrizable spaces, but what is the analog of Arhangel'skiĭ's definition? One possibility is to define “projective” as a continuous image of a projective subset of \mathbb{P} (or \mathbb{R}). We do not know whether this definition allows us to apply Projective Determinacy as desired. However, we do have the following.

Theorem 5.3. *The Continuum Hypothesis implies every productively Lindelöf, continuous image of a separable metrizable space is σ -compact.*

Proof. Let X be such an image. X is T_3 , X has a countable network and so X is separable and every closed subset is G_δ . The weight of X is $\leq 2^{\aleph_0}$, so by Alster's Lemma 2.6, X is Alster. But Alster spaces in which compact sets are G_δ are σ -compact [2]. \square

⁶ Technically, point-cofinite covers are required to be infinite. To see that this follows, note that if $\{U_n^k: k < \omega\}$ is finite, then there is U such that $U = U_{n_k}^k$ for infinitely many k . As X is not in \mathcal{U} , there is x which is not in U , and consequently not in infinitely many members of the sequence $\{U_n^k\}_{k < \omega}$, contradicting the assertion we are about to prove.

The *Baire Hierarchy* is formed by closing the collection of *closed* sets under countable unions and intersections. In contrast to the Borel Hierarchy, the Hurewicz Dichotomy fails at a low level. A $K_{\sigma\delta}$ space is a space which is the intersection of countably many σ -compact subspaces of some larger space.

Example 5.4. There is a $K_{\sigma\delta}$ space which is neither σ -compact nor includes a closed copy of \mathbb{P} .

Proof. In [7] Arhangel'skiĭ constructs a $K_{\sigma\delta}$ space, due to Okunev, which is not σ -compact but has only one non-isolated point, so does not include a closed copy of \mathbb{P} . The space is obtained by taking the Alexandrov duplicate of \mathbb{P} , and then collapsing the non-discrete copy of \mathbb{P} to a point. \square

5.1. κ -Analytic spaces

Descriptive set-theorist Ben Miller told us that the “right” definition of projective in a non-separable metrizable context is the following one.

Definition 5.5. A T_2 space X is κ -analytic, where κ is an uncountable cardinal, if X is a continuous image of the product of \aleph_0 copies of the discrete space of size κ .

Every space X is $|X|$ -analytic.

Recall that, according to the Hurewicz Dichotomy, every analytic non- σ -compact subspace of the Baire space contains a closed copy of the Baire space, and thus, if there is a Michael space, an analytic metrizable space is productively Lindelöf if and only if it is σ -compact. This and Example 5.4 motivate the following question.

Question 5.6. Let $\kappa < 2^{\aleph_0}$. Is it consistent that

- (1) Every non- σ -compact κ -analytic metrizable space includes a closed copy of \mathbb{P} ?
- (2) Every productively Lindelöf κ -analytic metrizable space is σ -compact?

It was L. Zdomskyy who pointed out to us that, if we drop the metrizability assumption, then the one-point Lindelöfication of the discrete space of size \aleph_1 gives a T_3 counter-example to both items of Question 5.6, and that $\kappa < 2^{\aleph_0}$ is necessary for the problem to have a possibly genuine descriptive set theoretic flavor.

The hypotheses in the following theorem, which answers (2) of Question 5.6, follow from Martin's Axiom plus the negation of the Continuum Hypothesis, see [3].

Theorem 5.7 (Zdomskyy). Assume that there is no cover of the Cantor space by \aleph_1 meager sets, and there is a Michael space. Then every productively Lindelöf \aleph_1 -analytic subset of the Cantor space is σ -compact.

Proof. The proof is analogous to that of Theorem 3.3 in [30]. The key is to observe that the proof of Repický's Theorem in [29], that Σ_2^1 subspaces of \mathbb{P} that are not the union of \aleph_1 compact subspaces include a closed copy of \mathbb{P} , also works for \aleph_1 -analytic subspaces. \square

Problem 5.8 (Zdomskyy). Is it consistent that every non- σ -compact space which is κ -analytic for some uncountable cardinal $\kappa < 2^{\aleph_0}$ includes a closed copy of \mathbb{P} ?

6. Spaces of countable type

In 1957, M. Henriksen and J.R. Isbell [20] introduced the class of (Tychonoff) spaces that are *Lindelöf at infinity*, i.e., the complement $\beta X \setminus X$ of the space X in its Stone–Čech compactification is Lindelöf. They proved that a Tychonoff space X is Lindelöf at infinity if and only if each compact subset of X is included in a compact $K \subseteq X$ such that $\chi(X, K) \leq \aleph_0$, i.e., there is a countable base for the neighborhoods of K in X . Arhangel'skiĭ [5] called spaces satisfying the latter equivalent condition of *countable type*. Locally compact spaces, metrizable spaces, Čech-complete spaces, and their common generalization, p -spaces, are all of countable type.

We present a simple proof for the following generalization of a result of Alster from [2].

Theorem 6.1. (Alas, et al. [1]) The Continuum Hypothesis implies every productively Lindelöf T_3 space of countable type and weight $\leq \aleph_1$ is σ -compact.

Proof. We generalize Michael's original proof that the Continuum Hypothesis implies productively Lindelöf metrizable spaces are σ -compact.

Embed X in $[0, 1]^{\aleph_1}$. Its closure in $[0, 1]^{\aleph_1}$ is a compactification γX of X . The identity map on X extends to a continuous surjection $f: \beta X \rightarrow \gamma X$, and since it fixes X , f maps $\beta X \setminus X$ onto $\gamma X \setminus X$ [15, 3.5.7]. As $\beta X \setminus X$ is Lindelöf, so is $\gamma X \setminus X$.

Assume that X is not σ -compact. Then $\gamma X \setminus X$ is not G_δ in γX . By the Continuum Hypothesis, we can take a collection $\{U_\alpha: \alpha < \aleph_1\}$ of open sets including $\gamma X \setminus X$, such that every open set including $\gamma X \setminus X$ includes some U_α . By taking countable intersections and thinning out, we can find a strictly decreasing sequence $\{V_\beta\}_{\beta < \aleph_1}$ of G_δ sets including $\gamma X \setminus X$, such that every open set including $\gamma X \setminus X$ includes some V_β . For each $\beta < \aleph_1$, take $p_\beta \in (V_{\beta+1} \setminus V_\beta) \cap X$.

Let

$$Y = (\gamma X \setminus X) \cup \{p_\beta: \beta < \aleph_1\}.$$

Put a topology on Y by strengthening the subspace topology to make each $\{p_\beta\}$ open. The usual Michael space argument [26] (cf. [1] for more general arguments) shows that Y is Lindelöf, but its product with X is not. Indeed, Y is Lindelöf, since each open set including $\gamma X \setminus X$ includes all but countably many p_β 's. To see that $Y \times X$ is not Lindelöf, note that the set $\{(p_\beta, p_\beta): \beta < \aleph_1\}$ is closed and discrete in $Y \times X$. \square

X is *absolute Borel* if it is Borel in βX . In this case, $\beta X \setminus X$ is a Baire subspace of βX . As Baire subspaces of compact T_2 spaces are Lindelöf [21], Lindelöf absolute Borel spaces are of countable type. We therefore have the following.

Corollary 6.2. *The Continuum Hypothesis implies productively Lindelöf absolute Borel spaces of weight $\leq \aleph_1$ are σ -compact.*

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Appendix A. Some remarks on elementary submodels and forcing

For the reader not so familiar with elementary submodels, we make some elementary remarks which may be helpful in understanding the proofs in this paper which involve this method.

First of all, the sets H_θ in our proofs appear only for technical reasons; we really think instead of the universe V . For elucidation of this point, see Chapter 24 of [25].

An elementary submodel M is *countably closed* if each countable subset of M is a member of M . For such models M , if $X \in M$, then the collection of countable sequences of members of $X \cap M$ is the same as the collection of countable sequences lying in M of members of X , i.e., $(X \cap M)^\omega = X^\omega \cap M$. A straightforward closing-off (Löwenheim–Skolem) argument establishes that H_θ , for regular $\theta \geq 2^{\aleph_0}$, has a countably closed elementary submodel of size 2^{\aleph_0} .

Definition A.1. For a topological space X with topology τ , X_M is the topological space $X \cap M$ with the topology with basis $\{U \cap M: U \in \tau \cap M\}$.

The proofs of the following basic facts are illustrative.

Lemma A.2 (Folklore). *Assume that X is a T_2 space, M is a countably closed elementary submodel of H_θ for some sufficiently large regular θ , and $X \in M$. Then: $X \cap M$ is a sequentially closed subset of X .*

Proof. Let $\{x_n\}_{n < \omega}$ be a sequence of elements of $X \cap M$ converging to a point $x \in X$. As M is countably closed and each $x_n \in M$, $\{x_n\}_{n < \omega} \in M$ as well. By elementarity, $M \models \{x_n\}_{n < \omega}$ converges to some point y . Since X is Hausdorff, $x = y \in M$, so $x \in X \cap M$. \square

Proposition A.3. (Junqueira and Tall [24]) *Assume that:*

- (a) X is a sequential T_2 space;
- (b) M is a countably closed elementary submodel of H_θ for some sufficiently large regular θ ; and
- (c) $X \in M$.

Then:

- (1) $X \cap M$ is a closed subset of X .

- (2) X_M is a continuous image of $X \cap M$.
 (3) For each property P of X preserved by continuous images and closed subspaces, X_M has the property P .

Proof. (1) follows from Lemma A.2, as X is sequential.

(2) The identity map from $X \cap M$ with the relative topology onto X_M is continuous, since every open set in X_M is open in $X \cap M$.

(3) follows from (1) and (2). \square

Another observation about countably closed models is that, roughly speaking, if properties involving countable sets (such as Lindelöfness) are true for the fragment of X lying in M , then M will demonstrate that, and thus, by elementarity, X will really have that property. Thus, such properties as powerfully Lindelöf, (finitely) powerfully Hurewicz, Menger, etc., go “up” from X_M to X . On the other hand, it is not so clear what happens with a property like *Alster*, since there can be expected to be compact subsets of X that are not in M .

We may, instead of going from a countably closed elementary submodel up to H_θ or the entire universe, go from the universe to an extension of it by countably closed forcing. A typical argument is then that if some property involving the existence of a countable object holds in the extension, it must have held in the original universe, since no new countable subsets of V were added by the forcing. Thus, if an open cover or sequence of open covers of X in V acquires some nice countable subcollection in the extension, it must have had that nice subcollection already. For example, if we find that X is Lindelöf, Menger, Hurewicz, etc., in a countably closed forcing extension, it must have had those properties to begin with. Again, a property such as *Alster* does not fit into this scheme, because countably closed forcing does not in general preserve compactness, and moreover can adjoin new compact sets.

Problem A.4. If X is σ -compact in a countably closed forcing extension, is it σ -compact?

The analogous problem is also open for countably closed elementary submodels.

Problem A.5. If M is a countably closed elementary submodel of H_θ for a sufficiently large regular θ with X and its topology as members, then if X_M is σ -compact, is X also?

For compactness, both problems have positive answers, and “countably closed” is not needed. This was noted earlier in the case of forcing; for elementary submodels, this was proved by L.R. Junqueira [23].

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